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Lax pair and Darboux transformation of a noncommutative $U(N)$ principal chiral model

U Saleem and M Hassan

Department of Physics, University of the Punjab, Quaid-e-Azam Campus, Lahore 54590, Pakistan

E-mail: usman_physics@yahoo.com and mhassan@physics.pu.edu.pk

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Abstract
We present a noncommutative generalization of the Lax formalism of the $U(N)$ principal chiral model in terms of a one-parameter family of flat connections. The Lax formalism is further used to derive a set of parametric noncommutative Bäcklund transformations and an infinite set of conserved quantities. From the Lax pair, we derive a noncommutative version of the Darboux transformation of the model.

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1. Introduction

During the last decade, there has been an increasing interest in the study of noncommutative field theories (nc-FTs) due to their relation to string theory, perturbative dynamics, quantum Hall effect, etc [1–16]. The noncommutative field theories can be constructed in different settings. One method of construction is through Moyal deformation product or $\star$-product (Moyal product) [17]. A simple noncommutative field theory (nc-FT) can be obtained by replacing the product of fields by their $\star$-product. The noncommutativity of coordinates of the Euclidean space R^D is defined as

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ is a second-rank antisymmetric real constant tensor known as a deformation parameter. It has been shown that in general, noncommutativity of time variables leads to non-unitarity and affects the causality of the theory [14, 15]. The $\star$-product of two functions in noncommutative Euclidean spaces is given by

$$(f \star g)(x) = f(x)g(x) + \frac{\mathrm{i}\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x) + \vartheta(\theta^2),$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. These nc-FTs reduce to the ordinary or commutative field theories (FTs) as the deformation parameter reduces to zero. There is also an increasing interest in the noncommutative extension of integrable field theories (nc-IFTs) [3–12]. Sometimes the noncommutativity breaks the integrability of a theory; however there are some examples in which integrability of a field theory is maintained [4]. In [4], a noncommutative extension of $U(N)$ principal chiral model (nc-PCM) has been presented and it is concluded that this noncommutative extension gives no extra constraints for the theory to be integrable. The non-local conserved quantities of nc-PCM have also been derived using the iterative method of Brézin–Itzykson–Zinn–Zuber (BIZZ) [18] but no effort has been made so far to study the Lax formalism of the nc-PCM and to derive conserved quantities and Darboux transformation from it.

In this paper we present a Lax formalism of one-parameter family of transformations on solutions of noncommutative $U(N)$ principal chiral model. The Lax formalism further gives a set of parametric noncommutative Bäcklund transformation (nc-BT) and a set of Riccati equations. The Lax formalism can be used to derive a series of conserved quantities. The Lax formalism is further used to develop the noncommutative version of Darboux transformation for the nc-PCM. We expand the Noether currents in power series in deformation parameter and obtain zeroth and first-order equations of motion and the conserved quantities.

2. Noncommutative principal chiral model

The action of $U(N)$ for the nc-PCM is defined by[1]

$$\mathcal{S}^\star = \frac{1}{2}\int \mathrm{d}^2x\, \mathrm{Tr}(\partial_+ g^{-1} \star \partial_- g), \tag{2.1}$$

with constraints on the fields $g(x^+, x^-)$:

$$g^{-1}(x^+, x^-) \star g(x^+, x^-) = g(x^+, x^-) \star g^{-1}(x^+, x^-) = 1,$$

where $g(x^+, x^-) \in U(N)$,[2] and $g^{-1}(x^+, x^-)$ stands for an inverse with respect to the $\star$-product.[3] The $U(N)$-valued field $g(x^+, x^-)$ is defined as

$$g(x^+, x^-) \equiv e_\star^{\mathrm{i}\pi_a T^a} = 1 + \mathrm{i}\pi_a T^a + \tfrac{1}{2}(\mathrm{i}\pi_a T^a)_\star^2 + \cdots,$$

where π_a is in the Lie algebra $u(N)$ of the Lie group $U(N)$ and T^a, $a = 1, 2, 3, \ldots, N^2$, are Hermitian matrices with the normalization $\mathrm{Tr}(T^a T^b) = -\delta^{ab}$ and are the generators of $U(N)$ in the fundamental representation satisfying the algebra

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c,$$

where f^{abc} are the structure constants of the Lie algebra $u(N)$. For any $X \in u(N)$, we write $X = X^a T^a$ and $X^a = -\mathrm{Tr}(T^a X)$.

The action (2.1) is invariant under a global continuous symmetry

$$U_L(N) \times U_R(N): \qquad g(x^+, x^-) \mapsto u \star g \star v^{-1}.$$

The associated Noether conserved currents of nc-PCM are

$$j_\pm^{\star R} = -g^{-1} \star \partial_\pm g, \qquad j_\pm^{\star L} = \partial_\pm g \star g^{-1},$$

[1] The nc-PCM can also be obtained by dimensional reduction of noncommutative anti-self dual Yang–Mills equations in four dimensions [13].

[2] Here we have taken the global symmetry group as $U(N)$, which has a simple noncommutative extension. There is no noncommutative $SU(N)$ because $\det(g_1 \star g_2) \neq \det(g_1) \star \det(g_2)$. Also for any $X, Y \in su(N)$ Lie algebra of the Lie group $SU(N)$, the commutator $X \star Y - Y \star X$ is not traceless. The noncommutative extensions of $SO(N)$ and $USp(N)$ have been constructed in [19–21] but the construction is a bit involved.

[3] Our conventions are such that the two-dimensional coordinates are related by $x^\pm = \frac{1}{2}(x^0 \pm \mathrm{i}x^1)$ and $\partial_\pm = \frac{1}{2}(\partial_0 \pm \mathrm{i}\partial_1)$.

which take values in the Lie algebra $u(N)$, so that one can decompose the currents into components $j^\star_\pm(x^+, x^-) = j^{\star a}_\pm(x^+, x^-)T^a$. The equation of motion following from (2.1) corresponds to conservation of these currents. The left and right currents satisfy the following conservation equation:

$$\partial_- j^\star_+ + \partial_+ j^\star_- = 0. \tag{2.2}$$

The currents also obey the zero-curvature condition

$$\partial_- j^\star_+ - \partial_+ j^\star_- + [j^\star_+, j^\star_-]_\star = 0, \tag{2.3}$$

where $[j^\star_+, j^\star_-]_\star = j^\star_+ \star j^\star_- - j^\star_- \star j^\star_+$. Equations (2.2) and (2.3) can also be expressed as

$$\partial_- j^\star_+ = -\partial_+ j^\star_- = -\tfrac{1}{2}[j^\star_+, j^\star_-]_\star. \tag{2.4}$$

Equations (2.2)–(2.4) hold for both $j^{\star L}_\pm$ and $j^{\star R}_\pm$.

3. Lax pair and conserved quantities of nc-PCM

In order to develop a Lax formalism and construct infinitely many conserved quantities for the nc-PCM, we define a one-parameter family of transformations on field $g(x^+, x^-)$ in the noncommutative space as

$$g \to g^{(\gamma)} = u^{(\gamma)} \star g \star v^{(\gamma)-1},$$

where γ is a parameter and $u^{(\gamma)}$, $v^{(\gamma)}$ are matrices belonging to $U(N)$. We choose the boundary values $u^{(1)} = 1$, $v^{(1)} = 1$ or $g^{(1)} = g$. The matrices $u^{(\gamma)}$ and $v^{(\gamma)}$ satisfy the following set of linear equations:

$$\partial_\pm u^{(\gamma)} = \tfrac{1}{2}(1 - \gamma^{\mp 1}) j^{\star L}_\pm \star u^{(\gamma)}, \tag{3.1}$$

$$\partial_\pm v^{(\gamma)} = \tfrac{1}{2}(1 - \gamma^{\mp 1}) j^{\star R}_\pm \star v^{(\gamma)}. \tag{3.2}$$

In what follows, we shall consider right-hand currents and drop the superscript R on the current to simply write $j^{\star R}_\pm = j^\star_\pm$. The compatibility condition of the linear system (3.2) is given by

$$\left\{(1 - \gamma^{-1})\partial_- j^\star_+ - (1 - \gamma)\partial_+ j^\star_- + \left(1 - \tfrac{1}{2}(\gamma + \gamma^{-1})\right)[j^\star_+, j^\star_-]_\star\right\} \star v^{(\gamma)} = 0.$$

Under the one-parameter family of transformations, the Noether-conserved currents transform as

$$j^\star_\pm \mapsto j^{\star(\gamma)}_\pm = \gamma^{\mp 1} v^{(\gamma)-1} \star j^\star_\pm \star v^{(\gamma)}.$$

The one-parameter family of conserved currents $j^{\star(\gamma)}_\pm$ in the noncommutative space for any value of γ : $\partial_+ j^{\star(\gamma)}_- + \partial_- j^{\star(\gamma)}_+ = 0$. The linear system (3.2) can be written as

$$\partial_\pm v(x^+, x^-; \lambda) = A^{\star(\lambda)}_\pm \star v(x^+, x^-; \lambda), \tag{3.3}$$

where the noncommutative fields $A^{\star(\lambda)}_\pm$ are given by

$$A^{\star(\lambda)}_\pm = \mp \frac{\lambda}{1 \mp \lambda} j^\star_\pm.$$

The parameter λ is a spectral parameter and is related to parameter γ by $\lambda = \frac{1-\gamma}{1+\gamma}$. The compatibility condition of the linear system (3.3) is the $\star$-zero-curvature condition

$$\left[\partial_+ - A^{\star(\lambda)}_+, \partial_- - A^{\star(\lambda)}_-\right]_\star \equiv \partial_- A^{\star(\lambda)}_+ - \partial_+ A^{\star(\lambda)}_- + \left[A^{\star(\lambda)}_+, A^{\star(\lambda)}_-\right]_\star = 0. \tag{3.4}$$

We have defined a one-parameter family of connections $A_\pm^{\star(\lambda)}$ which are flat. The Lax operators can now be defined as

$$L_\pm^{\star(\lambda)} = \partial_\pm - A_\pm^{\star(\lambda)}, \tag{3.5}$$

which obey the following equations:

$$\partial_\mp L_\pm^{\star(\lambda)} = \left[A_\mp^{\star(\lambda)}, L_\pm^{\star(\lambda)}\right]_\star. \tag{3.6}$$

The associated linear system (3.3) can be reexpressed as

$$\partial_0 v(x^0, x^1; \lambda) = A_0^{\star(\lambda)} \star v(x^0, x^1; \lambda), \qquad \partial_1 v(x^0, x^1; \lambda) = A_1^{\star(\lambda)} \star v(x^0, x^1; \lambda), \tag{3.7}$$

with the noncommutative connection fields $A_0^{\star(\lambda)}$ and $A_1^{\star(\lambda)}$ given by

$$A_0^{\star(\lambda)} = -\frac{\lambda}{1-\lambda^2}(j_1^\star + \lambda j_0^\star), \qquad A_1^{\star(\lambda)} = -\frac{\lambda}{1-\lambda^2}(j_0^\star + \lambda j_1^\star).$$

The compatibility condition of the linear system (3.7) is again the $\star$-zero-curvature condition for the fields $A_0^{\star(\lambda)}$ and $A_1^{\star(\lambda)}$:

$$\left[\partial_0 - A_0^{\star(\lambda)}, \partial_1 - A_1^{\star(\lambda)}\right]_\star \equiv \partial_1 A_0^{\star(\lambda)} - \partial_0 A_1^{\star(\lambda)} + \left[A_0^{\star(\lambda)}, A_1^{\star(\lambda)}\right]_\star = 0.$$

The Lax operator is defined as

$$L_1^{\star(\lambda)} = \partial_1 - A_1^{\star(\lambda)},$$

which obeys the Lax equation

$$\partial_0 L_1^{\star(\lambda)} = \left[A_0^{\star(\lambda)}, L_1^{\star(\lambda)}\right]_\star.$$

This equation gives the x^0-evolution of the operator $L_1^{\star(\lambda)}$ and is equivalent to an isospectral eigenvalue problem. We have been able to show that the existence of a one-parameter family of transformations and Lax formalism of PCM can be generalized to nc-PCM without any constraints. This works straightforwardly as it does in the commutative case. The one-parameter family of transformations thus gives rise to an infinite number of conserved quantities and the Darboux transformation of generating solution of nc-PCM.

3.1. Local conserved quantities

It is straight forward to derive an infinite set of local[4] conserved quantities from equation (2.4):

$$\partial_\mp \operatorname{Tr}(j_\pm^\star)^n = 0, \tag{3.8}$$

where n is an integer and the first case $n = 2$ corresponds to the conservation of the energy–momentum tensor. These conservation laws are associated with the totally symmetric invariant tensors of the Lie algebra $u(N)$ and the integers n turn out to be the exponents of $u(N)$. This is exactly what happens in the commutative case and these conserved quantities are shown to be in involution with each other [22], in the commutative case.

We can also derive the local conserved quantities of nc-PCM from the linear system (3.3) via noncommutative Bäcklund transformation (nc-BT) and Riccati equations. The linear system (3.3) reduces to the following set of noncommutative Bäcklund transformation (nc-BT):

$$\pm\partial_\pm(g^{-1} \star \tilde{g}) = \tilde{j}_\pm^\star - j_\pm^\star, \tag{3.9}$$

[4] The term 'local' in our discussion refers to its standard meaning. The conserved densities depend upon fields and their derivatives but not on their integrals. The intrinsic non-locality of the Moyal deformation products appearing due to the presence of derivatives to infinite order, persists in all our discussions. These conserved quantities are in fact deformed local conserved quantities carrying with them the intrinsic non-locality due to noncommutativity. Moreover, the leading terms in the perturbative expansion in θ are local.

with constraint $g^{-1}\star\tilde{g}+\tilde{g}^{-1}\star g = 2\lambda^{-1}I$, where λ is a real parameter, g and $\tilde{g}$ are solutions of equation of motion. The nc-BT given by equation (3.9) further gives rise to a set of compatible noncommutative Riccati equations

$$\partial_{\pm}\Gamma(\lambda) = -\frac{\lambda}{2(1\mp\lambda)}\left(j_{\pm}^{\star}+\Gamma(\lambda)\star j_{\pm}^{\star}\star\Gamma(\lambda)-2\lambda^{-1}j_{\pm}^{\star}\star\Gamma(\lambda)\mp[\Gamma(\lambda), j_{\pm}^{\star}]_{\star}\right), \tag{3.10}$$

where $\Gamma(\lambda) = g^{-1}\star\tilde{g}$. The linearization of the Riccati equation (3.10) gives rise to the linear system (3.3). Equations (2.2), (2.3) and (3.10) can be used to give a series of conservation laws:

$$(1+\lambda)\partial_{-}\operatorname{Tr}(\Gamma(\lambda)\star j_{+}^{\star}) - (1-\lambda)\partial_{+}\operatorname{Tr}(\Gamma(\lambda)\star j_{-}^{\star}) = 0. \tag{3.11}$$

Expanding $\Gamma(\lambda)$ as a power series in λ: $\Gamma(\lambda) = \sum_{k=0}^{\infty}\lambda^{k}\Gamma_{k}$, one can generate λ-independent conservation laws. It is not easier to solve the algebraic equations obtained recursively from (3.11) when we substitute the expansion of $\Gamma(\lambda)$. The explicit form of conservation law is therefore not quite transparent. It is, therefore, not straightforward to relate these local conserved quantities with the ones associated with invariant tensors of $u(N)$.

The existence of nontrivial higher spin local conserved quantities has important implications regarding classical and quantum integrability of a field theory. In the two-dimensional quantum field theory, their existence forces the multiparticle scattering matrix to factorize into a product of two particle scattering matrices and eventually to be computed exactly. The two particle S-matrix satisfies the Yang–Baxter equation [23–25]. We expect that the local conserved quantities in nc-PCM will also give some important information about the complex dynamics of the model. The higher spin local conserved quantities of the type (3.8) are related to the W-algebra structure appearing in certain conformal field theories. The deformed local conserved quantities would naturally lead to the study of deformation of W-algebra [26].

3.2. Non-local conserved quantities

An infinite number of non-local[5] conserved quantities can also be generated from the Lax formalism of nc-PCM. We assume spatial boundary conditions such that the currents $j^{\star(\gamma)}$ vanish as $x^{1}\to\pm\infty$. Equation (3.7) implies that $v(x^{0},\infty;\lambda)$ are time independent. The residual freedom in the solution for $v(x^{0},\infty;\lambda)$ allows us to fix $v(x^{0},\infty;\lambda) = 1$, the unit matrix and we are then left with the x^{0}-independent matrix-valued function

$$Q^{\star}(\lambda) = v(x^{0},\infty;\lambda). \tag{3.12}$$

Expanding $Q^{\star}(\lambda)$ as power series in λ gives an infinite number of non-local conserved quantities

$$Q^{\star}(\lambda) = \sum_{k=o}^{\infty}\lambda^{k}Q^{\star(k)}, \qquad \frac{\mathrm{d}}{\mathrm{d}x^{0}}Q^{\star(k)} = 0. \tag{3.13}$$

For the explicit expressions of the non-local conserved quantities, we write (3.7) as

$$v(x^{0},x^{1};\lambda) = 1-\frac{\lambda}{1-\lambda^{2}}\int_{-\infty}^{x^{1}}\mathrm{d}y\,(j_{0}^{\star}(x^{0},y)-\lambda j_{1}^{\star}(x^{0},y))\star v(x^{0},y;\lambda). \tag{3.14}$$

We expand the field $v(x^{0},x^{1};\lambda)$ as power series in λ as

$$v(x^{0},x^{1};\lambda) = \sum_{k=o}^{\infty}\lambda^{k}v_{k}(x^{0},x^{1}), \tag{3.15}$$

[5] Here again the term 'non-local' refers to the meaning that the conserved densities depend on fields, their derivatives and their integrals and they also contain intrinsic non-locality of the Moyal deformation.

and compare the coefficients of powers of λ, we get a series of conserved non-local currents, which upon integration give non-local conserved quantities. The first two non-local conserved quantities of nc-PCM are

$$Q^{\star(1)} = -\int_{-\infty}^{\infty} \mathrm{d}y\, j_0^\star(x^0, y),$$
$$Q^{\star(2)} = -\int_{-\infty}^{\infty} \mathrm{d}y\, j_1^\star(x^0, y) + \int_{-\infty}^{\infty} \mathrm{d}y\, j_0^\star(x^0, y) \star \int_{-\infty}^{y} \mathrm{d}z\, j_0^\star(x^0, z).$$

These conserved quantities are exactly the same as obtained in [4] using the noncommutative iterative method of Brezin *et al* [18]. We now show that the procedure outlined above is equivalent to the iterative construction of non-local conserved quantities of nc-PCM [4]. From equations (3.3) and (3.15), we get

$$\partial_\pm \sum_{k=o}^{\infty} \lambda^k v_k(x^0, x^1) = \pm D_\pm \sum_{k=o}^{\infty} \lambda^k v_k(x^0, x^1),$$

where the covariant derivatives $D_\pm$ are defined as

$$D_\pm v^{(k)} = \partial_\pm v^{(k)} - j_\pm^\star \star v^{(k)} \qquad \Rightarrow \qquad [D_+, D_-]_\star = 0.$$

We can now define currents $j_\pm^{\star(k)}$ for $k = 0, 1, \ldots$ which are conserved in the noncommutative space such that

$$\partial_- j_+^{\star(k)} + \partial_+ j_-^{\star(k)} = 0, \qquad \Leftrightarrow \qquad j_\pm^{\star(k)} = \pm\partial_\pm v^{(k)}.$$

An infinite sequence of conserved non-local currents can be obtained by iteration[6]

$$j_\pm^{\star(k+1)} = D_\pm v^{(k)} \qquad \Rightarrow \qquad \partial_- j_+^{\star(k+1)} + \partial_+ j_-^{\star(k+1)} = 0.$$

This establishes the equivalence of noncommutative Lax formalism and noncommutative iterative construction. Here we have been able to use nc-Lax formalism of nc-PCM to generate an infinite sequence of non-local conserved quantities and have been able to relate them to the nc-iterative procedure.

Let us make a few comments about the algebra of these conserved quantities. In the commutative case, the local conserved quantities based on the invariant tensors, all Poisson commute with each other and with the non-local conserved quantities. The classical Poisson brackets of non-local conserved quantities constitute classical Yangian symmetry $Y(u(N))$[27]. The Yangian is related to the Yang–Baxter equation [28] showing the consistency with the factorization of multiparticle S-matrix. The fundamental irreducible representations of the Yangian correspond to particle multiplets and tensor product rules of Yangian correspond to the Dorey's fusing rules [29]. In the noncommutative case, we expect that noncommutative Yangian appears in the model and its quantum version can shed some light on the nonperturbative behaviour of the model. One way of investigating the algebra of non-local conserved quantities is to develop a canonical formalism in the noncommutative space and to derive noncommutative Poisson bracket current algebra of the model. In this work we have not attempted to answer these questions and will return to these issues in some later work.

[6] The non-local conserved currents for the noncommutative models can also be constructed by using bidifferential calculi and Hodge decomposition of the differential forms for elements in the algebra of the noncommutative torus [8–10].

4. Darboux transformation of nc-PCM

The Lax pair of nc-PCM obtained in the previous section can be further used to define Darboux transformation of generating solutions of the linear system (3.2) of nc-PCM. We follow the procedure of constructing Darboux transformation of PCM adopted in [30]. For convenience, we write the linear system (3.2) as

$$\begin{aligned} \partial_+ v(x^+, x^-, \mu) &= \mu(2\mu - 1)^{-1} j_+^\star \star v(x^+, x^-, \mu), \\ \partial_- v(x^+, x^-, \mu) &= \mu j_-^\star \star v(x^+, x^-, \mu), \end{aligned} \tag{4.1}$$

where $\mu = \frac{1-\gamma}{2}$ and $v(x^+, x^-, \mu)$ is a non-degenerate $N \times N$ fundamental matrix solution of system (4.1). The currents $j_+^\star$ and $j_-^\star$ obey the following condition[7]:

$$j_+^\star + j_-^{\star\dagger} = 0. \tag{4.2}$$

Equations (2.2) and (2.3) can be written as the compatibility condition of the linear system (4.1), that is,

$$\mu(\partial_- j_+^\star - \partial_+ j_-^\star + [j_+^\star, j_-^\star]_\star) + (\mu - 1)(\partial_- j_+^\star + \partial_+ j_-^\star) = 0.$$

In order to construct a noncommutative version of Darboux transformation, we proceed as follows. $v[1]$ be another matrix solution of the linear system (4.1). The onefold Darboux transformation relates the solutions $v[1]$ and v by the following equation:

$$v[1] = D(\mu) \star v, \tag{4.3}$$

where $D(\mu)$,

$$D(\mu) = I - \mu S, \tag{4.4}$$

is the Darboux matrix and $S(x^+, x^-)$ is the $N \times N$ matrix function and I is the identity matrix. The linear system for $v[1]$ is given by

$$\begin{aligned} \partial_+ v[1] &= \mu(2\mu - 1)^{-1} j_+^\star[1] \star v[1], \\ \partial_- v[1] &= \mu j_-^\star[1] \star v[1], \end{aligned} \tag{4.5}$$

where $j_+^\star[1]$ and $j_-^\star[1]$ satisfy equations (2.2) and (2.3). Applying $\partial_\pm$ on equation (4.3) and equating the coefficients of different powers of μ, we get the following equations:

$$j_+^\star[1] = j_+^\star + \partial_+ S, \qquad j_-^\star[1] = j_-^\star - \partial_- S, \tag{4.6}$$

and

$$\begin{aligned} \partial_+ S \star S - 2\partial_+ S &\equiv S \star j_+^\star - j_+^\star \star S = [S, j_+^\star]_\star, \\ \partial_- S \star S &\equiv j_-^\star \star S - S \star j_-^\star = -[S, j_-^\star]_\star. \end{aligned} \tag{4.7}$$

One can solve equation (4.7) to get $S(x^+, x^-)$ so that $j_+^\star[1]$, $j_-^\star[1]$ and $v[1]$ are obtained from (4.6), (4.3) and (4.4) respectively. An explicit expression for the matrix $S(x^+, x^-)$ can be found as follows.

Let us take N complex numbers $\mu_1, \mu_2, \ldots, \mu_N$ ($\neq 0, 1/2$) which are not all same. Also take N constant column vectors $w_1, w_2, \ldots, w_N$ and construct a non-degenerate $N \times N$ matrix

$$M = (v(\mu_1)w_1, v(\mu_2)w_2, \ldots, v(\mu_N)w_N), \tag{4.8}$$

with $\det M \neq 0$. Each column $m_\alpha = v(\mu_\alpha)w_\alpha$ in the matrix M is a solution of the linear system (4.1) for $\mu = \mu_\alpha$, i.e.

$$\partial_+ m_\alpha = \mu_\alpha(2\mu_\alpha - 1)^{-1} j_+^\star \star m_\alpha, \qquad \partial_- m_\alpha = \mu_\alpha j_-^\star \star m_\alpha, \tag{4.9}$$

[7] The $U(N)$ group is composed of all $N \times N$ matrices. Then for all $g \in U(N)$, $g^\dagger = g^{-1}$ where $g^\dagger$ is the Hermitian conjugate of g. A matrix $P \in u(N)$ Lie algebra of $U(N)$ if and only if $P^\dagger + P = 0$.

where $\alpha = 1, 2, \ldots, N$. The matrix form of equations (4.9) will be

$$\partial_+ M = j_+^\star \star M\Lambda(2\Lambda - 1)^{-1}, \qquad \partial_- M = j_-^\star \star M\Lambda. \tag{4.10}$$

Let us take the matrix

$$\Lambda = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_N) \tag{4.11}$$

such that the matrix

$$S = M \star \Lambda^{-1} \star M^{-1}, \tag{4.12}$$

satisfies equation (4.7).

Our next step is to check that equation (4.12) is a solution of equation (4.7). In order to show this, we first apply ∂_+ on equation (4.12) to get

$$\partial_+ S = (j_+^\star \star S - S \star j_+^\star) \star M \star \Lambda(2\Lambda - 1)^{-1} \star M^{-1},$$

or

$$\partial_+ S \star S - 2\partial_+ S \equiv S \star j_+^\star - j_+^\star \star S = [S, j_+^\star]_\star.$$

Similarly, we apply ∂_- on equation (4.12) to get

$$\partial_- S = j_-^\star - S \star j_-^\star \star S^{-1},$$

or

$$\partial_- S \star S \equiv j_-^\star \star S - S \star j_-^\star = -[S, j_-^\star]_\star.$$

This shows that equation (4.12) is a solution of equation (4.7). Equations (4.3), (4.4) and (4.6) define a Darboux transformation for the nc-PCM. In order to have $j_+^\star[1], j_-^\star[1] \in u(N)$ we need to show that

$$\partial_\pm (S - S^\dagger) = 0. \tag{4.13}$$

In other words we want to make specific S to satisfy (4.13). This can be achieved if we choose

$$\mu_\alpha = \begin{cases} \rho_1(\alpha = 1, 2, \ldots, k) \quad (0 < k < N) \\ \rho_2(\alpha = k+1, k+2, \ldots, N). \end{cases} \tag{4.14}$$

Now take ρ_1 to be an imaginary number and define

$$\rho_2 = \frac{\bar{\rho}_1}{2\bar{\rho}_1 - 1}, \tag{4.15}$$

with $|2\rho_1 - 1| \neq 1$ so that $\rho_1 \neq \rho_2$. This has been defined for the later convenience. Now we define column solutions for eigenvalues ρ_1 and ρ_2. Let $m_1, m_2, \ldots, m_k$ and $m_{k+1}, m_{k+2}, \ldots, m_N$ be the column solutions of the linear system (4.1) for $\mu = \rho_1$ and $\mu = \rho_2$ respectively, i.e.

$$\begin{aligned} &m_p = v(\rho_1) w_p, \qquad m_q = v(\rho_2) w_q \\ &(p = 1, 2, \ldots, k, q = k+1, k+2, \ldots, N). \end{aligned} \tag{4.16}$$

We have to choose w_α so that

$$m_q^\dagger \star m_p = 0, \qquad (p = 1, 2, \ldots, k, q = k+1, k+2, \ldots, N) \tag{4.17}$$

at one point (say $(0, 0)$) and m_α are linearly independent. We shall show that the matrix S constructed from these values of μ_α and w_α satisfies (4.13).

First, we prove that (4.17) holds everywhere if it holds at one point. The proof of the above identity (4.17) is as follows. Let us first calculate

$$
\begin{aligned}
\partial_-\left(m_q^\dagger \star m_p\right) &= \partial_- m_q^\dagger \star m_p + m_q^\dagger \star \partial_- m_p \\
&= (\partial_+ m_q)^\dagger \star m_p + m_q^\dagger \star \partial_- m_p, \\
&= \left(\frac{\rho_2}{2\rho_2 - 1} j_+^\star \star m_q\right)^\dagger \star m_p + m_q^\dagger \star \partial_- m_p, \\
&= -\left(\frac{\bar{\rho}_2}{2\bar{\rho}_2 - 1}\right) m_q^\dagger \star j_-^\star \star m_p + m_q^\dagger \star \partial_- m_p, \\
&= -\rho_1 m_q^\dagger \star j_-^\star \star m_p + \rho_1 m_q^\dagger \star j_-^\star \star m_p, \\
&= 0.
\end{aligned}
$$

Similarly we can check

$$\partial_+\left(m_q^\dagger \star m_p\right) = 0.$$

This means that (4.17) holds everywhere if it holds at one point, i.e.

$$\partial_\pm\left(m_q^\dagger \star m_p\right) = 0. \tag{4.18}$$

The m_α's are linearly independent everywhere if they are linearly independent at one point as (4.9) is linear. We have to choose m_α so that they are linearly independent and (4.18) holds everywhere.

From equation (4.12) we have

$$S \star m_p = \frac{1}{\rho_1} m_p, \qquad S \star m_q = \frac{1}{\rho_2} m_q. \tag{4.19}$$

The Hermitian conjugate of equation (4.19) is given by

$$m_p^\dagger \star S^\dagger = \frac{1}{\bar{\rho}_1} m_p^\dagger, \qquad m_q^\dagger \star S^\dagger = \frac{1}{\bar{\rho}_2} m_q^\dagger. \tag{4.20}$$

Therefore

$$
\begin{aligned}
m_p^\dagger \star (S^\dagger - S) \star m_r &= \left(\frac{1}{\bar{\rho}_1} - \frac{1}{\rho_1}\right) m_p^\dagger \star m_r, \\
m_q^\dagger \star (S^\dagger - S) \star m_s &= \left(\frac{1}{\bar{\rho}_2} - \frac{1}{\rho_2}\right) m_q^\dagger \star m_s
\end{aligned} \tag{4.21}
$$

or

$$
\begin{aligned}
&m_q^\dagger \star (S^\dagger - S) \star m_p = 0, \\
&m_p^\dagger \star (S^\dagger - S) \star m_q = 0 \qquad (p, r = 1, 2, \ldots, k, q, s = k+1, k+2, \ldots, N).
\end{aligned} \tag{4.22}
$$

From equation (4.15) we have

$$\frac{1}{\rho_1} - \frac{1}{\bar{\rho}_1} = \frac{1}{\rho_2} - \frac{1}{\bar{\rho}_2}. \tag{4.23}$$

Take

$$m_\beta^\dagger \star (S^\dagger - S) \star m_\alpha = m_\beta^\dagger \star \left(\frac{1}{\bar{\rho}_1} - \frac{1}{\rho_1}\right) \star I \star m_\alpha, \qquad (\alpha, \beta = 1, 2, \ldots, N) \tag{4.24}$$

and $\frac{1}{\rho_1} - \frac{1}{\rho_2}$ is real. Since the set $\{m_\alpha\}$ consists of N linearly independent vectors

$$S^\dagger - S = \left(\frac{1}{\bar{\rho}_1} - \frac{1}{\rho_1}\right) I, \tag{4.25}$$

therefore equation (4.25) implies that

$$j_+^\star[1] + j_-^{\star\dagger}[1] = j_+^\star + j_-^{\star\dagger} - \partial_+(S^\dagger - S) = 0. \tag{4.26}$$

This proves that $j_+^\star[1]$ and $j_-^{\star\dagger}[1]$ satisfy equation (4.2) for $U(N)$. To summarize our results, we write the onefold Darboux transformation as

$$(j_+^\star, j_-^\star, v) \longrightarrow (j_+^\star[1], j_-^\star[1], v[1]),$$

where

$$v[1] = (I - \mu S)v, \qquad j_+^\star[1] = j_+^\star + \partial_+ S, \qquad j_-^\star[1] = j_-^\star - \partial_- S,$$

and $v[1]$ is the solution of the following linear system:

$$\partial_+ v[1] = \mu(2\mu - 1)^{-1} j_+^\star[1] \star v[1], \qquad \partial_- v[1] = \mu j_-^\star[1] \star v[1],$$

such that the matrix S satisfies the following equations:

$$\partial_+ S \star S - 2\partial_+ S = [S, j_+^\star]_\star, \qquad \partial_- S \star S = -[S, j_-^\star]_\star,$$

and the currents $j_+^\star[1]$, $j_-^\star[1]$ satisfy the following condition:

$$j_+^\star[1] + j_-^{\star\dagger}[1] = 0.$$

The twofold Darboux transformation is

$$(j_+^\star[1], j_-^\star[1], v[1]) \longrightarrow (j_+^\star[2], j_-^\star[2], v[2]),$$

where

$$v[2] = (I - \mu S[1])v[1],$$
$$j_+^\star[2] = j_+^\star[1] + \partial_+ S[1], \qquad j_-^\star[2] = j_-^\star[1] - \partial_- S[1].$$

and $v[2]$ is the solution of the following linear system:

$$\partial_+ v[2] = \mu(2\mu - 1)^{-1} j_+^\star[2] \star v[2], \qquad \partial_- v[2] = \mu j_-^\star[2] \star v[2],$$

such that the matrix $S[1]$ satisfies the following equations:

$$\partial_+ S[1] \star S[1] - 2\partial_+ S[1] = [S[1], j_+^\star[1]]_\star, \qquad \partial_- S[1] \star S[1] = -[S[1], j_-^\star[1]]_\star,$$

and the matrix $S[1]$ is given by

$$S[1] = M[1] \star \Lambda^{-1} \star M[1]^{-1},$$

with $M[1]$ obeying

$$\partial_+ M[1] = j_+^\star[1] \star M[1]\Lambda(2\Lambda - 1)^{-1}, \qquad \partial_- M[1] = j_-^\star[1] \star M[1]\Lambda,$$

and the currents $j_+^\star[2]$, $j_-^\star[2]$ satisfy the following condition:

$$j_+^\star[2] + j_-^{\star\dagger}[2] = 0.$$

The result can be generalized to obtain K-fold Darboux transformation

$$(j_+^\star[K-1], j_-^\star[K-1], v[K-1]) \longrightarrow (j_+^\star[K], j_-^\star[K], v[K]),$$

where

$$v[K] = (I - \mu S[K-1])v[K-1],$$
$$j_+^\star[K] = j_+^\star[K-1] + \partial_+ S[K-1], \qquad j_-^\star[K] = j_-^\star[K-1] - \partial_- S[K-1],$$

and $v[K]$ is the solution of

$$\partial_+ v[K] = \mu(2\mu - 1)^{-1} j_+^\star[K] \star v[K], \qquad \partial_- v[K] = \mu j_-^\star[K] \star v[K],$$

such that the matrix $S[K-1]$ satisfies the following equations:

$$\partial_+ S[K-1] \star S[K-1] - 2\partial_+ S[K-1] = [S[K-1], j_+^\star[K-1]]_\star,$$
$$\partial_- S[K-1] \star S[K-1] = -[S[K-1], j_-^\star[K-1]]_\star.$$

The matrix $S[K-1]$ is given by

$$S[K-1] = M[K-1] \star \Lambda^{-1} \star M[K-1]^{-1},$$

such that $M[K-1]$ obeys

$$\partial_+ M[K-1] = j_+^\star[K-1] \star M[K-1]\Lambda(2\Lambda-1)^{-1},$$
$$\partial_- M[K-1] = j_-^\star[K-1] \star M[K-1]\Lambda,$$

where the currents $j_+^\star[K]$, $j_-^\star[K]$ satisfy the following condition:

$$j_+^\star[K] + j_-^{\star\dagger}[K] = 0.$$

This completes the iteration of Darboux transformation. From a given seed solution one can generate the noncommutative multi-soliton solutions of the system. Such solutions have been constructed for the noncommutative integrable $U(N)$ sigma model in 2 + 1 dimensions by employing the dressing method and their scattering properties have been investigated [31, 32]. In addition, these multi-soliton solutions correspond to D0-branes moving inside the D2-branes in the open $N = 2$ fermionic string theory [33, 34]. For two-dimensional Euclidean sigma models, the noncommutative multi-solitons and their moduli space have been constructed that unifies different descriptions of Abelian and non-Abelian multi-solitons [35]. For nc-PCM such solutions can be explicitly constructed either by Darboux transformation or by the dressing method. One can also generalize the uniton method [36, 37] of constructing solutions for nc-PCM. These non-trivial solutions of nc-PCM are presented for a given projection operator in [38] where the construction of the unitons for nc-PCM is based on the noncommutative generalization of the theorem due to Uhlenbeck [36, 37]. We shall return to the complete description of the construction of uniton solutions of nc-PCM and proof of noncommutative version of the theorem of Uhlenbeck in some later work.

5. Perturbative expansion

In this section we study the perturbative expansion of the noncommutative fields of nc-PCM and compute the equation of motion and the conserved quantities up to first order in perturbative expansion in noncommutativity parameter θ. We can expand the currents $j_\pm$ as power series in θ. We expand the currents $j_\pm$ up to first order in θ:

$$j_\pm^\star = j_\pm^{[0]} + \theta j_\pm^{[1]}. \tag{5.1}$$

By substituting the value of $j_\pm^\star$ from equation (5.1) into equation of motion (2.2) and (2.3), we get

$$\begin{aligned}
&\partial_- j_+^{[0]} + \partial_+ j_-^{[0]} = 0,\\
&\partial_- j_+^{[1]} + \partial_+ j_-^{[1]} = 0,\\
&\partial_- j_+^{[0]} - \partial_+ j_-^{[0]} + \left[j_+^{[0]}, j_-^{[0]}\right] = 0,\\
&\partial_- j_+^{[1]} - \partial_+ j_-^{[1]} + \left[j_+^{[1]}, j_-^{[0]}\right] + \left[j_+^{[0]}, j_-^{[1]}\right] = -\frac{\mathrm{i}}{2}\left(j_{++}^{[0]} j_{--}^{[0]} + j_{--}^{[0]} j_{++}^{[0]}\right) - \frac{\mathrm{i}}{8}\left[j_+^{[0]}, j_-^{[0]}\right]^2,
\end{aligned} \tag{5.2}$$

where $j_{\pm\pm}^{[0]} = \partial_\pm j_\pm^{[0]}$. It is clear from the above equations that the currents $j_\pm^{[0]}$ and $j_\pm^{[1]}$ are conserved, $j_\pm^{[0]}$ is curvature free but $j_\pm^{[1]}$ is not curvature free.

The perturbative expansion of iterative construction gives the following results:

$$\begin{aligned} j_\pm^{[0](k+1)} &= D_\pm^{[0]} v^{[0](k)}, \quad \Rightarrow \quad \partial_- j_+^{[0](k+1)} + \partial_+ j_-^{[0](k+1)} = 0, \\ j_\pm^{[1](k+1)} &= D_\pm^{[0]} v^{[1](k)} - D_\pm^{[1]} v^{[0](k)}, \quad \Rightarrow \quad \partial_- j_+^{[1](k+1)} + \partial_+ j_-^{[1](k+1)} = 0, \end{aligned}$$

where

$$\begin{aligned} D_\pm^{[0]} v^{[0,1](k)} &= \partial_\pm v^{[0,1](k)} - j_\pm^{[0]} v^{[0,1](k)}, \\ D_\pm^{[1]} v^{[0](k)} &= j_\pm^{[1]} v^{[0](k)} + \frac{\mathrm{i}}{2}\left(\partial_\pm j_\pm^{[0]} \partial_\mp - \partial_\mp j_\pm^{[0]} \partial_\pm\right) v^{[0](k)}. \end{aligned}$$

From this analysis, it is obvious that the conservation of kth current implies the conservation of $(k+1)$ th current at zeroth as well as first order of perturbation expansion in the parameter of noncommutativity.

By substituting the value of $j_\pm^\star$ from equation (5.1) into equation (3.8), we obtain first four zeroth and first-order local conserved quantities

$$\begin{aligned} &\partial_\mp \operatorname{Tr}\left(j_\pm^{[0]2}\right) = 0, \\ &\partial_\mp \operatorname{Tr}\left(j_\pm^{[0]} j_\pm^{[1]}\right) = 0, \\ &\partial_\mp \operatorname{Tr}\left(j_\pm^{[0]3}\right) = 0, \\ &\partial_\mp \operatorname{Tr}\left(j_\pm^{[0]2} j_\pm^{[1]} - \tfrac{1}{4} j_\pm^{[0]} j_{\pm\pm}^{[0]}\left[j_\pm^{[0]}, j_\mp^{[0]}\right] + \tfrac{1}{4} j_{\pm\pm}^{[0]} j_\pm^{[0]}\left[j_\pm^{[0]}, j_\mp^{[0]}\right]\right) = 0, \\ &\partial_\mp \operatorname{Tr}\left(j_\pm^{[0]4}\right) = 0, \\ &\partial_\mp \operatorname{Tr}\left(j_\pm^{[0]3} j_\pm^{[1]} - \tfrac{1}{8} j_\pm^{[0]2} j_{\pm\pm}^{[0]}\left[j_\pm^{[0]}, j_\mp^{[0]}\right] + \tfrac{1}{8} j_{\pm\pm}^{[0]} j_\pm^{[0]2}\left[j_\pm^{[0]}, j_\mp^{[0]}\right]\right) = 0. \end{aligned}$$

The conservation laws hold because of equation (5.2). The conserved holomorphic currents $\partial_\mp \operatorname{Tr}\left(j_\pm^{[0]}\right)^2, \partial_\mp \operatorname{Tr}\left(j_\pm^{[0]}\right)^3, \partial_\mp \operatorname{Tr}\left(j_\pm^{[0]}\right)^4, \ldots,$ are the usual local currents and the corresponding local conserved quantities are

$$Q_{\pm s}^{[0]} = \int_{-\infty}^{\infty} \mathrm{d}x \operatorname{Tr}\left(j_\pm^{[0]}\right)^n,$$

where $s = n - 1$ represents the spin of the conserved quantity. The higher spin conserved quantities are in involution with each other, i.e.

$$\begin{aligned} &\left\{Q_{+s}^{[0]}, Q_{-r}^{[0]}\right\} = 0, \qquad r, s > 0 \\ &\left\{Q_{\pm s}^{[0]}, Q_{\pm r}^{[0]}\right\} = 0. \end{aligned}$$

The values of s are precisely the exponents modulo the Coxeter number of Lie algebra $u(N)$. These conservation laws are also related to the symmetric invariant tensors of the $u(N)$ and the zeroth-order contributions give the commuting conserved quantities with spins equal to the exponents of the underlying algebra. We also expect that the first-order conserved quantities are also in involution and the calculations involve the Poisson bracket current algebra of the model which we have not been able to find at this stage.

Similarly we can expand conserved quantities

$$\begin{aligned} Q^{(1)[0]} &= -\int_{-\infty}^{\infty} j_0^{[0]}(x^0, y)\, \mathrm{d}y, \\ Q^{(1)[1]} &= -\int_{-\infty}^{\infty} j_0^{[1]}(x^0, y)\, \mathrm{d}y, \\ Q^{(2)[0]} &= \int_{-\infty}^{\infty} \left(-j_1^{[0]}(x^0, y) + j_0^{[0]}(x^0, y) \int_{-\infty}^{y} j_0^{[0]}(x^0, z)\, \mathrm{d}z\right) \mathrm{d}y, \\ Q^{(2)[1]} &= \int_{-\infty}^{\infty} \left(-j_1^{[1]}(x^0, y) + j_0^{[1]}(x^0, y) \int_{-\infty}^{y} j_0^{[0]}(x^0, z)\, \mathrm{d}z + j_0^{[0]}(x^0, y) \int_{-\infty}^{y} j_0^{[1]}(x^0, z)\, \mathrm{d}z\right) \mathrm{d}y. \end{aligned}$$

The conservation of these quantities can be proved by using equation (5.2). The non-local conserved quantities $Q^{(1)[0]}$ and $Q^{(2)[0]}$ form the usual Yangian $Y(u(N))$. There are two copies of this structure corresponding to right and left currents and therefore the algebra is $Y_L(u(N)) \times Y_R(u(N))$. These zeroth-order Yangian conserved quantities also Poisson commutate with the zeroth-order local conserved quantities $Q_{\pm s}^{[0]}$ i.e.

$$\left\{Q_{\pm s}^{[0]}, Q^{(1)[0]}\right\} = 0, \qquad \left\{Q_{\pm s}^{[0]}, Q^{(2)[0]}\right\} = 0.$$

The first-order contribution in the algebra of both local and non-local conserved quantities can be investigated if the deformed canonical Poisson bracket algebra of deformed currents is known. Note that the first-order correction to the first non-local conserved quantity is an integral of non-local function of the fields. These corrections shall naturally modify the Yangian structure of the non-local conserved quantities and as a result a Moyal-deformed Yangian might appear, whose zeroth-order element must be the usual Yangian of the given Lie algebra.

6. Conclusions

In this paper, we have analysed the Lax formalism of nc-PCM. In this generalization, we have observed that noncommutative extension works straightforwardly resulting in a noncommutative equation of PCM without any constraint appearing due to noncommutativity. The Lax formalism of nc-PCM has been used to generate local as well as non-local conserved quantities of the model and it has been shown that the Lax formalism of nc-PCM is equivalent to the iterative procedure already used in [4]. Furthermore, the Lax formulism has been used to derive K-fold Darboux transformation of the nc-PCM. The noncommutative Darboux transformation can be used to construct non-trivial solutions of the nc-PCM and to study their moduli-space dynamics. The present work can be extended to construct the uniton solutions of nc-PCM and to investigate the algebra of local and non-local conserved quantities. Another interesting direction to pursue is to look at the quantum conservation of the local and non-local conserved quantities of the nc-PCM. The method of anomaly counting for the quantum mechanical survival of the local conservation laws can also be applied to nc-PCM [27–29]. For the non-local conserved quantities the quantum Yangians can also be investigated for the nc-PCM [39, 40]. It is also interesting to seek noncommutative extension of the Lax formalism, local and non-local conserved quantities of supersymmetric PCM in the direction adopted in [41, 42] for the commutative supersymmetric PCM.

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